

SUBMERSIONS, HAMILTONIAN SYSTEMS AND OPTIMAL SOLUTIONS TO THE ROLLING MANIFOLDS PROBLEM

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ABSTRACT. Given a submersion $\pi : Q \rightarrow M$ with an Ehresmann connection \mathcal{H} , we describe how to solve Hamiltonian systems on M by lifting our problem to Q and doing our computations here. We also show how solutions of lifted Hamiltonian systems on Q can be viewed as particles in M under the influence of a generalization of the Lorentz force. We investigate what this means in terms of normal and abnormal extremals in optimal control problems on respectively M and Q . In particular, we are interested in submersions between sub-Riemannian manifolds. We apply our theory to a concrete example, namely solving the problem of how we can roll one Riemannian manifold M on another \widehat{M} without twisting or slipping along curves of minimal length, connecting two given configurations. Here, we both use a submersion to lift our problem to a space where computations are simpler to execute, and we show that the curves in each of the manifolds can be considered as trajectories of objects with a non-constant gauge-charge under the influence of the Lorentz force. The Yang-Mills field is in this case given by the difference of the Riemannian curvatures of the manifolds M and \widehat{M} . In particular, we show that in the case of two surfaces rolling on each other, solutions are determined by the equation of a pendulum whose length is inverse proportional to the difference in Gaussian curvature of the two surfaces.

1. INTRODUCTION

Consider a submersion $\pi : Q \rightarrow M$ between two Riemannian manifolds satisfying the following condition. Let \mathcal{V} be the subbundle of TQ formed by the kernel of the differential π_* and use \mathcal{H} for its orthogonal complement. Then π is called a Riemannian submersion if π_* is an isometry on each fiber when restricted to \mathcal{H} . Several papers exist connecting the geometry of Q with M , see e.g. [8, 9, 16]. One of the possibilities such a submersion gives us, is that we are able to find the geodesics of M , by looking for geodesics in Q which are horizontal to \mathcal{H} at one point (and hence all points) and then projecting them to M . This can be a great advantage if it is easier to do computations on the space Q than it is on M . A simple example of this, is the case the case when Q is a Lie group and M is a symmetric space.

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A similar idea was used in [12], for the case when the top space is a sub-Riemannian manifold. For any Ehresmann connection \mathcal{H} on a submersion $\pi : Q \rightarrow M$ into a Riemannian manifold M , if we give Q a sub-Riemannian structure by lifting the metric of M to \mathcal{H} , then geodesics in M lift to normal geodesics in Q . For the definition of sub-Riemannian manifolds and normal geodesics, see Section 3.3. Both Riemannian geodesics and normal sub-Riemannian geodesics may be considered as solutions to Hamiltonian systems. We will show that given any Hamiltonian function H on M and an a choice on Ehresmann connection \mathcal{H} on the submersion $\pi : Q \rightarrow M$, we can define a lifted Hamiltonian function \tilde{H} on Q . We then describe how we can obtain the solutions of H on M by looking at some of the solutions of \tilde{H} on Q . Using this result, we are able to solve any Hamiltonian system on M by doing computations on Q .

We also show a converse result inspired by the following example. Consider a principle G -bundle $\pi : Q \rightarrow M$, where the base manifold M is a Riemannian manifold. Again, let \mathcal{H} be an Ehresmann connection on π , invariant under the action of G , and lift the metric of M to \mathcal{H} , giving Q the structure of a sub-Riemannian manifold. In this case, projections the normal geodesics of Q are the trajectories of gauge-charged particles under the influence of the Lorentz force [13]. Here, the gauge is an element in the dual Lie algebra of G and the Yang Mills field is represented by the curvature of \mathcal{H} . See Section 3.4 for details. Generalizing this idea, we show how solutions of lifted Hamiltonian systems on Q can be viewed from the base space M . These projections are curves following a combination of the Hamiltonian vector field of the original Hamiltonian, along with a “force” depending on the curvature of \mathcal{H} . The concept of gauge is replaced by an introduction of parallel transport for forms which vanish on \mathcal{H} . By doing these modification, we are able to describe what way solutions are influenced by the original Hamiltonian and the connection \mathcal{H} , respectively.

The paper is structured as follows. In Section 2 we describe how to lift a Hamiltonian system from the base space to the top space of a submersion. Then we present our main theorem, connecting solutions of the two Hamiltonian system. In Section 3 we apply this result to optimal control problems, in particular sub-Riemannian manifolds. Given a submersion $\pi : Q \rightarrow M$, we both describe how to solve an optimal control problem given on M by doing computations on Q , and how solutions of lifted optimal control problems on Q look when viewed from M .

In Section 4 we give a specific example to show the effectiveness of our approach. We look at the problem of finding the optimal curves of the kinematic system of two Riemannian manifolds M and \tilde{M} rolling on each other under the constraint of high friction, preventing slipping and twisting. Here, results has, up until now, only been presented in some specific cases [10, 11], and even the case of surfaces with non constant curvature rolling on each other was unknown. We present the equations solving this optimal control

problem using exactly the techniques described above. First, we lift the problem from our original configuration space to the product of the orthogonal frame bundles $F(M) \times F(\widehat{M})$, where it is simpler to do computations. Then, we describe the solutions by looking at the curve projected to one of the manifolds. We find that these projected curves behave as objects with an $\mathfrak{so}(n)^*$ -gauge under the influence of the Lorentz force, where this gauge changes according to a given vector field. The Yang-Mills field is in this case given by the differences in the Riemannian curvatures. In particular, we show in the case of two surfaces rolling on each other without twisting or slipping, the solution of this problem are in many cases found by solving the equation of a pendulum, whose length is inverse proportional to the differences in Gaussian curvature of M and \widehat{M} along the curve.

Some of the proofs, including the proof of the main theorem, are left to Section 5.

2. LIFTING HAMILTONIAN SYSTEMS

2.1. Notation and conventions. For any vector bundle $\Pi^E : E \rightarrow M$, we will use $\Gamma(E)$ is the space of smooth sections of E . Rather than writing $X(m)$ for the value of the section $X : M \rightarrow E$ at $m \in M$, we will use the notation $X|_m$. The zero section of the vector bundle will be denoted by $\mathbf{0}^E$. If E is a subbundle of the tangent bundle, then a curve γ in M is called E -horizontal if it is absolutely continuous and has its tangent vector in E for almost every t . For elements $e_1, e_2 \in E_m$ in the fiber over the same element $m \in M$, define the vector $\text{vl}_{e_1} e_2 \in T_{e_1} E$ by

$$\text{vl}_{e_1} e_2 = \left. \frac{d}{dt} \right|_{t=0} (e_1 + te_2).$$

This is called the vertical lift of e_2 to e_1 . Similarly, for any $X \in \Gamma(E)$, we can define the vertical lift $\text{vl } X \in \Gamma(T E)$ of X by $\text{vl } X|_e = \text{vl}_e X|_{\Pi^E(e)}$.

Rather than introducing a special notation for the interior product, we will for example denote the interior product of a two-form η by a vector field X by $\eta(X, \cdot)$. Its pullback by a map f will be denoted $\eta(X, f_* \cdot)$.

2.2. Ehresmann connections, curvature and parallel transport. Let $\pi : Q \rightarrow M$ be a submersion between two connected manifolds Q and M . By a submersion, we mean a surjective map such that the differential map π_* is surjective as well. We assume that the manifolds Q and M have respective dimensions $n + \nu$ and n . For each $m \in M$, we write $Q_m = \pi^{-1}(m)$ for the preimage of m . In what follows, objects related to Q will typically be marked by a tilde (\sim).

We call the subbundle $\mathcal{V} = \ker \pi_*$ the vertical space. An Ehresmann connection \mathcal{H} on $\pi : Q \rightarrow M$ is a subbundle of TQ satisfying $TQ = \mathcal{H} \oplus \mathcal{V}$. Since for every $q \in Q$, the map $\pi_*|_{\mathcal{H}_q} : \mathcal{H}_q \rightarrow T_{\pi(q)} M$ is a linear isomorphism, a choice of Ehresmann connection allows us to define horizontal lifts of vectors, vector fields and curves.

To be more specific, for every vector $v \in T_m M$ and $q \in Q_m$, we define the horizontal lift $h_q v$ of v to q as the unique element in \mathcal{H}_q which is sent to v by π_* . Similarly, any vector field $X \in \Gamma(TM)$ has a horizontal lift hX given by formula

$$hX|_q := h_q X|_{\pi(q)}.$$

Finally, a curve $\tilde{\gamma}$ in Q is a horizontal lift of an absolutely continuous curve γ in M if $\tilde{\gamma}$ is \mathcal{H} -horizontal and satisfy $\pi(\tilde{\gamma}) = \gamma$. In other words, $\tilde{\gamma}$ is a solution of the equation $\dot{\tilde{\gamma}} = h_{\tilde{\gamma}} \dot{\gamma}$. Clearly this implies that $\tilde{\gamma}$ is uniquely determined by γ up to initial condition. For a general submersion $\pi : Q \rightarrow M$ and curve γ in M , we only know that a horizontal lift exist for short time.

The curvature operator corresponding to \mathcal{H} is defined as

$$\tilde{\mathcal{R}}(\tilde{X}, \tilde{Y}) = \text{pr}_{\mathcal{V}} \left[\text{pr}_{\mathcal{H}} \tilde{X}, \text{pr}_{\mathcal{H}} \tilde{Y} \right] \quad \text{for any } \tilde{X}, \tilde{Y} \in \Gamma(TQ).$$

Here $\text{pr}_{\mathcal{V}}$ and $\text{pr}_{\mathcal{H}}$ are respective projections to \mathcal{H} and \mathcal{V} , having the other bundle as their kernel. This gives us an operator $\mathcal{R} : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(\mathcal{V})$ defined as $\mathcal{R}(X, Y) = \tilde{\mathcal{R}}(hX, hY)$. It is simple to verify that \mathcal{R} is $C^\infty(M)$ -linear in both arguments.

We also want to introduce an idea of parallel transport of sections in \mathcal{V} . We do this by defining an operator

$$\begin{aligned} \blacktriangledown : \Gamma(TM) \times \Gamma(\mathcal{V}) &\rightarrow \Gamma(\mathcal{V}) \\ (X, \Upsilon) &\mapsto \blacktriangledown_X \Upsilon := [hX, \Upsilon] \end{aligned}$$

This is \mathbb{R} -linear in the second argument and $C^\infty(M)$ -linear in the first argument. Hence $\blacktriangledown_X \Upsilon|_q$ only depends on $X|_{\pi(q)}$ in the first argument.

Lemma 1. *Let $\gamma : [0, t_1] \rightarrow M$ be an absolutely continuous curve with $\gamma(0) = m_0$. Let q_0 be a given point in Q_{m_0} , and let $\tilde{\gamma}$ be the horizontal lift of γ with $\tilde{\gamma}(0) = q_0$. Then for any $\Upsilon_0 \in \mathcal{V}_{q_0}$, there is a unique vertical vector field $\Upsilon(t)$ along $\tilde{\gamma}$ with $\blacktriangledown_{\dot{\tilde{\gamma}}(t)} \Upsilon(t) = 0$ and $\Upsilon(0) = \Upsilon_0$.*

Proof. Let (x, U) be a chart on M around m_0 , and chose a neighborhood \tilde{U} in $\pi^{-1}(U)$ around q_0 sufficiently small such that $\mathcal{V}|_{\tilde{U}}$ is trivial. Let $\Upsilon_1, \dots, \Upsilon_\nu$ be sections of \mathcal{V} , forming a basis on \tilde{U} . Write

$$\blacktriangledown_{h\partial_{x_i}} \Upsilon_\kappa = \sum_{\mu=1}^{\nu} \Gamma_{i\kappa}^\mu \Upsilon_\mu,$$

and $\dot{\gamma}(t) = \sum_{i=1}^n \dot{x}_i(t) \partial_{x_i}$. Then $\Upsilon(t) = \sum_{\kappa=1}^{\nu} b_\kappa(t) \Upsilon_\kappa|_{\tilde{\gamma}(t)}$ is a solution to

$$\dot{\tilde{\gamma}} = \sum_{i=1}^n \dot{x}_i h\partial_{x_i}, \quad 0 = \blacktriangledown_{\dot{\tilde{\gamma}}} \Upsilon = \sum_{\kappa=1}^{\nu} \left(\dot{b}_\kappa + \sum_{i=1}^n \sum_{\mu=1}^{\nu} \dot{x}_i b_\mu \Gamma_{i\kappa}^\mu \right) \Upsilon_\kappa|_{\tilde{\gamma}}.$$

These equations along with the initial conditions uniquely determines Υ . \square

Remark 1. In contrary to parallel transport with respect to an affine connection in a vector bundle, we are only sure that we are able to define parallel

transport for short time, since the horizontal lift of γ may not be defined for all time.

2.3. Induced submersion of the cotangent bundle and lifted Hamiltonian system. Given a choice of splitting $TQ = \mathcal{H} \oplus \mathcal{V}$ of TQ , we will also have a corresponding splitting of the cotangent bundle $T^*Q = \text{Ann}(\mathcal{V}) \oplus \text{Ann}(\mathcal{H})$. By $\text{Ann}(\mathcal{V})$ and $\text{Ann}(\mathcal{H})$ we mean elements of T^*Q which vanish on respectively \mathcal{V} or \mathcal{H} . Note that for each $q \in Q$, we have a linear isomorphism $T_{\pi(q)}^*M \rightarrow \text{Ann}(\mathcal{V})_q$ given by $p \mapsto \pi^*(p)$. Corresponding to our choice of subbundle \mathcal{H} , we have a unique way to extend the inverse of this map to a submersion of the entire cotangent bundle.

Let us introduce the vector bundle map

$$\begin{array}{ccc} T^*Q & \xrightarrow{\pi^2} & T^*M \\ \Pi^{T^*Q} \downarrow & & \downarrow \Pi^{T^*M} \\ Q & \xrightarrow{\pi} & M \end{array},$$

uniquely defined by the properties $\ker \pi^2 = \text{Ann}(\mathcal{H})$ and $\pi^2(\tilde{p})(v) = \tilde{p}(h_q v)$ for any $q \in Q$, $\tilde{p} \in T_q^*Q$ and $v \in T_{\pi(q)}^*M$.

With this formalism, we are now ready to present our main theorem. Let $H \in C^\infty(T^*M)$ be a Hamiltonian function, and use \vec{H} for its Hamiltonian vector field. By abusing terminology slightly, we will often refer to an integral curve $\lambda(t)$ of \vec{H} as simply *an integral curve of H* .

Theorem 1. *Let $H \in C^\infty(T^*M)$ and define $\tilde{H} \in C^\infty(T^*Q)$ as $\tilde{H} = H \circ \pi^2$.*

- (a) *A curve in $\lambda(t)$ is an integral curve of H if and only if it is, at least for short time, the projection of an integral curve $\tilde{\lambda}(t)$ of \tilde{H} which is contained in $\text{Ann}(\mathcal{V})$ (it is sufficient to require this in only one point).*
- (b) *Let $\tilde{\lambda} : [0, t_1] \rightarrow T^*Q$ be an integral curve of \tilde{H} with $\tilde{\gamma}(t) := \Pi^{T^*Q}(\tilde{\lambda}(t))$ and $v_0 := \tilde{\lambda}(0) \text{pr}_{\mathcal{V}}$. Let $\pi^2(\tilde{\lambda}) = \lambda$ with $\gamma = \Pi^{T^*M}(\lambda)$.*

Then λ is a solution to

$$(1) \quad \dot{\lambda}(t) = \vec{H}|_{\lambda(t)} - \text{vl}_{\lambda(t)} v(t) \mathcal{R}(\dot{\gamma}(t), \cdot)$$

where the curve $v(t)$ in $\text{Ann}(\mathcal{H})$ determined by equations

$$(2) \quad \nabla_{\dot{\gamma}(t)} v(t) = 0, \quad v(0) = v_0.$$

The curve $\tilde{\gamma}$ can be obtained again as an \mathcal{H} -horizontal lift of γ .

In the above theorem, $v(t) \mathcal{R}(\dot{\gamma}(t), \cdot) \in T_{\gamma(t)}^*M$ is the element such that for any $Y_0 \in T_{\gamma(t)}M$ it takes the value $v(t) \mathcal{R}(X, Y)$, where X and Y are any vector fields with $X|_{\gamma(t)} = \dot{\gamma}(t)$ and $Y|_{\gamma(t)} = Y_0$. Parallel transport in (2) is defined such that for any $X \in \Gamma(TM)$, $\Upsilon \in \Gamma(\mathcal{V})$ and $\alpha \in \Gamma(\text{Ann}(\mathcal{H}))$, we have $(\nabla_X \alpha)(\Upsilon) = X(\alpha(\Upsilon)) - \alpha(\nabla_X \Upsilon)$.

The proof of this theorem is left to Section 5.1.

Remark 2. In this presentation, we started by choosing an Ehresmann \mathcal{H} and used this to construct π^2 and \tilde{H} . Conversely, suppose that we have a Hamiltonian \tilde{H} which is equal to $H \circ \pi^2$ for some $H \in C(T^*M)$ and vector bundle morphism $\pi^2 : T^*Q \rightarrow T^*M$ over π which is a left inverse to π^* on every fiber. Then we can define \mathcal{H} as the vectors annihilated by $\ker \pi^2$, and the same result holds.

2.4. Special case: Hamiltonian of a Riemannian manifold. We now consider the following special case. Let (M, \mathbf{g}) be a Riemannian manifold with metric \mathbf{g} . Associated to this metric, let $\flat : TM \rightarrow T^*M$ be the bundle isomorphism $v \mapsto \mathbf{g}(v, \cdot)$ with inverse \sharp . Use ∇ for the Levi-Civita connection corresponding to \mathbf{g} . Associated to the Riemannian structure, we also have a Hamiltonian function $H(p) = \frac{1}{2}p(\sharp p)$. The projection of integral curves of H to M gives us the Riemannian geodesics. If we apply Theorem 1(b) to this Hamiltonian we get the following result.

Corollary 1. *Let $\pi : Q \rightarrow M$ be a submersion into a Riemannian manifold (M, \mathbf{g}) , with associated Hamiltonian H . Let $\mathcal{H} \subseteq TQ$ be a chosen Ehresmann connection, and define \tilde{H} as in Section 2.3.*

*Let $\tilde{\lambda} : [0, t_1] \rightarrow T^*Q$ be an integral curve of \tilde{H} with $\tilde{\lambda}(0) \text{pr}_V = v_0$. Use $\tilde{\gamma}$ for its projection to Q . Then $\tilde{\gamma}$ is a horizontal lift of a curve γ in M , which is a solution of*

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = -\sharp v(t) \mathcal{R}(\dot{\gamma}, \cdot),$$

where $\nabla_{\dot{\gamma}} v(t) = 0$ and $v(0) = v_0$.

Proof. Pick a local basis e_1, \dots, e_n be a local basis orthonormal vector fields, and use these vector field to give the cotangent bundle local coordinates $p_i = p(e_i)$. It is standard result that the Hamiltonian vector field in these local coordinates is given by

$$\vec{H} = \sum_{i=1}^n p_i e_i - \sum_{i,j,k=1}^n p_j p_k \Gamma_{jk}^i \partial_{p_i}, \quad \Gamma_{ij}^k = \mathbf{g}(e_k, \nabla_{e_i} e_j).$$

Let $\lambda(t) = \pi^2(\tilde{\lambda}(t))$. If $\lambda(t) = \sum_{i=1}^n p_i(t) \flat e_i|_{\gamma(t)}$ is a solution to

$$\dot{\lambda}(t) = \vec{H}|_{\lambda(t)} - \text{vl}_{\lambda(t)} v(t) \mathcal{R}(\dot{\gamma}(t), \cdot),$$

then $\flat e_i(\dot{\gamma}(t)) = p_i(t)$ and $\dot{p}_i(t) = -\sum_{j,k=1}^n p_j(t) p_k(t) \Gamma_{jk}^i|_{\gamma(t)} - v(t) \mathcal{R}(\dot{\gamma}(t), e_i)$. The corollary then follows from the computation

$$\begin{aligned} \nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) &= \sum_{i=1}^n \dot{p}_i(t) e_i + \sum_{i,j,k=1}^n p_i p_j \Gamma_{ij}^k e_k \\ &= \sum_{i=1}^n \left(-\sum_{j,k=1}^n p_j p_k \Gamma_{jk}^i - v(t) \mathcal{R}(\dot{\gamma}(t), e_i) \right) e_i + \sum_{i,j,k=1}^n p_i p_j \Gamma_{ij}^k e_k \\ &= -\sharp v(t) \mathcal{R}(\dot{\gamma}(t), \cdot). \end{aligned}$$

□

3. LIFTED OPTIMAL CONTROL PROBLEMS

3.1. Optimal control systems. There are many different definitions and generalizations of an optimal control problem. We will use the definition found in [3, Section 2.1] and [2].

A smooth control system consists of a fiber bundle $\xi : \mathcal{U} \rightarrow M$ with fiber U , along with a bundle morphism

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{f} & TM \\ & \searrow \xi & \swarrow \Pi^{TM} \\ & M & \end{array}$$

A measurable essentially bound curve $\kappa(t)$ in \mathcal{U} is called *an admissible control* if its projection $\gamma(t)$ in M is an L^∞ curve satisfying $\dot{\gamma} = f(\kappa(t))$. We want to consider the following optimal control problem: For a smooth function $\varphi : C^\infty(\mathcal{U}) \rightarrow \mathbb{R}$ and two points $m_0, m_1 \in M$, find the admissible control $\kappa(t)$ which satisfies

$$\gamma(0) = m_0, \quad \gamma(t_1) = m_1, \quad \text{with} \quad \xi(\kappa(t)) = \gamma(t),$$

and minimize the functional

$$\kappa \mapsto \int_0^{t_1} \varphi(\kappa(t)) dt.$$

A sufficient condition for an admissible control to be a solution to this optimal control problem is given by the Pontryagin Maximum Principle (PMP). In order to present this result in a simpler way, we will just write the formulation assuming that \mathcal{U} can be written as a trivial fiber bundle $\mathcal{U} = M \times U$, which can be considered a local version of the general case. See Remark 3 for how this statement can be reformulated for the case when \mathcal{U} cannot be trivialized.

Theorem 2. PMP for Optimal Control Problem with fixed time t_1 [1, Theorem 12.3]:

Let $\hat{\kappa}(t) = (\hat{\gamma}(t), \hat{u}(t))$ be a solution to the above optimal control problem, where $\hat{\gamma}(t)$ is a curve in M and $\hat{u}(t)$ is a curve in U . For each $r \in \mathbb{R}, u \in U$, consider a Hamiltonian function

$$(3) \quad H_{r,u}(p) = p(f(m, u)) + r\varphi(m, u), \quad p \in T_m^*M.$$

Then there exists a curve $\lambda : [0, t_1] \rightarrow T^*M$, and a number $r \leq 0$ such that

- (i) $\Pi^{T^*M}(\lambda(t)) = \hat{\gamma}(t)$.
- (ii) $\dot{\lambda}(t) = \vec{H}_{r, \hat{u}(t)}|_{\lambda(t)}$ for almost every t ,
- (iii) $H_{r, \hat{u}(t)}(\lambda(t)) = \max_{u \in U} H_{r,u}(\lambda(t))$ for almost every t .

Moreover, if $r = 0$, then λ never intersects the zero section of T^*M .

Let us use the name *extremals* for solutions of PMP. They are called normal if $r \neq 0$ (it is sufficient to consider $r = -1$) and abnormal if $r = 0$.

Remark that abnormal extremals do not depend on the function φ , only the control system (ξ, f) .

Remark 3. If the fiber bundle cannot be trivialized, PMP can be reformulated in the following way. For any $r \in \mathbb{R}$, define $\mathcal{H}_r : \mathcal{U} \times_M T^*M \rightarrow \mathbb{R}$ by $\mathcal{H}_r(\kappa, p) = p(f(\kappa)) + r\varphi(\kappa)$, which takes the place of the Hamiltonian in (3). Requirement (ii) is then replaced with the identity

$$(\text{pr}_2^* \varsigma)|_{\tilde{\kappa}(t), \lambda(t)}(\tilde{X}, \dot{\lambda}) = (\mathcal{H}_r)_*|_{(\tilde{\kappa}(t), \lambda(t))}(\tilde{X}).$$

for any vector field $\tilde{X} \in \Gamma(T(\mathcal{U} \times_M T^*M))$, where $\text{pr}_2 : \mathcal{U} \times_M T^*M \rightarrow T^*M$ is the projection and ς is the canonical symplectic form on T^*M . In the requirement (iii), the maximum needs to hold over all elements in $\mathcal{U}_{\tilde{\gamma}(t)}$.

3.2. Submersions and a lifted optimal control problem. Let the pair $\xi : \mathcal{U} \rightarrow M$ and $f : \mathcal{U} \rightarrow TM$ be a control system and let $\pi : Q \rightarrow M$ be a submersion. Then a choice of Ehresmann connection \mathcal{H} on π gives us a unique way to induce a control system on Q .

- Define a fiber bundle $\tilde{\xi} : \pi^* \mathcal{U} \rightarrow Q$ as the pull-back of $\xi : \mathcal{U} \rightarrow M$. Recall that this is defined as

$$\pi^* \mathcal{U} = \{(q, \kappa) \in Q \times \mathcal{U} : \pi(q) = \xi(\kappa)\}.$$

The map $\tilde{\xi}$ is just the projection on the first coordinate. Let $\tilde{\pi}$ denote the projection on the second coordinate. Then we have the following commutative diagram

$$\begin{array}{ccc} \pi^* \mathcal{U} & \xrightarrow{\tilde{\pi}} & \mathcal{U} \\ \tilde{\xi} \downarrow & & \downarrow \xi \\ Q & \xrightarrow{\pi} & M \end{array}$$

- Define a bundle morphism $\tilde{f} : \pi^* \mathcal{U} \rightarrow TQ$ by

$$\tilde{f}(\tilde{\kappa}) = h_{\tilde{\xi}(\tilde{\kappa})} f(\tilde{\pi}(\tilde{\kappa})) \quad \text{for any } \tilde{\kappa} \in \pi^* \mathcal{U}.$$

In other words, if $\tilde{\kappa}$ represents the pair (q, κ) , then $\tilde{f}(\tilde{\kappa}) = h_q f(\kappa)$.

On this control system, we consider the optimal control problem induced by the function $\tilde{\varphi} = \varphi \circ \tilde{\pi}$. In this situation, extremals of Q and M , both normal and abnormal, are connected through Theorem 1. It is sufficient to show this locally, so we assume that we can identify $\pi^* \mathcal{U}$ with $Q \times U$. We then verify that for any $\tilde{p} \in T^*Q$ and $u \in U$, we have

$$\begin{aligned} \tilde{H}_{r,u}(\tilde{p}) &:= \tilde{p}(\tilde{f}(q, u)) + r\tilde{\varphi}(q, u), \\ &= \tilde{p}(h_q f(\pi(q), u)) + r\varphi(\pi(q), u) \\ &= \pi^2(\tilde{p})(f(\pi(q), u)) + r\varphi(\pi(q), u) = H_{r,u}(\pi^2(\tilde{p})). \end{aligned}$$

This relation can be a powerful and useful tool in many optimal control problems. We will explore the case of sub-Riemannian manifolds in detail.

3.3. Submersions and sub-Riemannian manifolds. A sub-Riemannian manifold is a triple (M, D, \mathbf{g}) , where M is a connected manifold, D is a sub-bundle of TM and \mathbf{g} is a metric tensor on D . Use AC_D for the Hilbert manifold of all D -horizontal curves defined on the interval $[0, 1]$ with an L^2 derivative. Let $AC_D(m_0, m_1)$ be the subset of AC_D consisting of curves γ satisfying $\gamma(0) = m_0$ and $\gamma(1) = m_1$. The distance function d_{cc} in this geometry is called the Carnot-Carathéodory distance and is defined as

$$d_{cc}(m_0, m_1) = \inf \left\{ \int_0^1 \mathbf{g}(\dot{\gamma}, \dot{\gamma})^{1/2} dt : \gamma \in AC_D(m_0, m_1) \right\}.$$

A sufficient condition for this distance to be finite, i.e. that any pair of points can be connected by a D -horizontal curve, is that D is bracket generating. This means that D along with the iterated brackets of its sections span the entire tangent bundle. To look for minimizers with respect to this distance function, is equivalent to look for curves in $AC_D(m_0, m_1)$ which minimize the energy functional

$$E(\gamma) = \frac{1}{2} \int_0^1 \mathbf{g}(\dot{\gamma}, \dot{\gamma}) dt.$$

This can be viewed as the control system with data (Π^D, inc) , such that $\Pi^D : D \rightarrow M$ is the projection and $\text{inc} : D \rightarrow TM$ is just the inclusion, where we try to minimize $E(\gamma)$. The only difference is that we allow the derivatives to be in L^2 rather than L^∞ .¹

Normal extremals can be described in the following way. The metric tensor \mathbf{g} defines a bundle map $\sharp_{\mathbf{g}} : T^*M \rightarrow D$ by the identity $\mathbf{g}(\sharp_{\mathbf{g}}p, v) = p(v)$ for any $v \in D$. Normal extremals are then the solution of the Hamiltonian system with Hamiltonian $H_{sR}(p) = \frac{1}{2}p(\sharp_{\mathbf{g}}p)$. Projections to M of normal extremals are always smooth and are length minimizers locally. We therefore call such curves in M *normal geodesics*. Remark that if λ is a normal extremal, and its projection to M is the normal geodesic γ , then we have the relation

$$(4) \quad \dot{\gamma}(t) = \sharp_{\mathbf{g}}\lambda(t)$$

Abnormal extremals λ also have an alternate description. Let ς be the canonical symplectic form on T^*M and let λ be a curve in T^*M . Then λ is an abnormal extremal if and only if it is an absolutely continuous curve in $\text{Ann}(D)$ with an L^2 derivative, never meeting the zero section, such that $\varsigma(\dot{\lambda}(t), \cdot)|_{\text{Ann}(D)} = 0$. A curve fulfilling the latter requirement, is often referred to as a *characteristic* of D . We will use the term *an abnormal curve* $\gamma(t)$ if the curve is the projection to M of an abnormal extremal $\lambda(t)$. Abnormal curves are singular points of the map from the Hilbert manifold $AC_D(m_0)$ of curves in AC_D starting at m_0 to the manifold M given by

$$\text{end} : AC_D(m_0) \rightarrow M, \quad \text{end}(\gamma) = \gamma(1).$$

¹Some authors prefer to work with sub-Riemannian geometry, requiring that horizontal curves should have L^∞ derivatives.

For more details on sub-Riemannian manifolds, see [14]

Let $\pi : Q \rightarrow M$ be a submersion, with vertical space \mathcal{V} and a chosen Ehresmann connection \mathcal{H} . We lift the sub-Riemannian structure in the same way that we do with a more general optimal control problems. The lifted structure can be described as the sub-Riemannian manifold $(Q, \tilde{D}, \tilde{\mathbf{g}})$, where

$$\tilde{D} = \{h_q v : v \in D_m, Q \in Q_m, m \in M\} = (\pi_*)^{-1}(D) \cap \mathcal{H},$$

and $\tilde{\mathbf{g}} = \pi^* \mathbf{g}$. We will say that this sub-Riemannian structure on Q is lifted from M . We present the following corollaries of Theorem 1.

Corollary 2. *A curve $\lambda(t)$ in T^*M is a normal extremal if and only if it is a projection of a normal extremal in T^*Q contained in $\text{Ann}(\mathcal{V})$. Conversely, projections of normal extremals in T^*Q satisfy equation (1) with $H = H_{sR}$.*

Remark 4. In the special case when $D = TM$, making the base space a Riemannian manifold, projections of normal geodesics are given by Corollary 1. The top space is then a sub-Riemannian manifold $(Q, \mathcal{H}, \pi^* \mathbf{g})$. Also, \mathcal{H} -horizontal lifts of Riemannian geodesics are normal geodesics. The latter fact was first observed in [12, Th. 6.2, Cor 6.5]

Theorem 3. *A curve γ in M is abnormal if and only if any horizontal lift of the curve is abnormal. Conversely, an L^2 -curve $\tilde{\lambda}(t)$ in T^*Q with $\pi^2(\tilde{\lambda}(t)) = \lambda(t)$, $\Pi^{T^*Q}(\tilde{\lambda}(t)) = \tilde{\gamma}(t)$, $v(t) = \lambda(t) \text{pr}_{\mathcal{V}}$ and $\Pi^{T^*M}(\lambda(t)) = \gamma(t)$, is an abnormal extremal if and only if $\tilde{\gamma}$ is \tilde{D} -horizontal,*

$$\varsigma(\dot{\lambda}, \cdot)|_{\text{Ann}(\tilde{D})} = -v(t) \mathcal{R}(\dot{\gamma}, \Pi_*^{T^*M} \cdot),$$

$$\nabla_{\dot{\gamma}} v = 0,$$

and λ and v does not vanish simultaneously.

The proof of this theorem uses elements of the proof of Theorem 1, and is therefore left to Section 5.2.

3.4. Special case: Charged particles in a magnetic field or a Yang-Mills field. Consider the special case when the submersion $\pi : Q \rightarrow M$ is a principal G -bundle, where G acts by a right action. Corresponding to this action, we have vector fields $\sigma(A)$ defined such that for any A in the Lie algebra \mathfrak{g} of G ,

$$(5) \quad \sigma(A)|_q = \left. \frac{d}{dt} \right|_{t=0} q \cdot \exp_G(tA).$$

For every q , the map $\mathfrak{g} \mapsto \mathcal{V}|_q$ defined by $A \mapsto \sigma(A)|_q$ is a linear isomorphism.

Let \mathcal{H} be an Ehresmann connection which is invariant under the group action, that is, $\mathcal{H}_p \cdot a = \mathcal{H}_{p \cdot a}$ for any $a \in G$. Such an Ehresmann connection is called *principal*, and corresponding to it is a principal connection form ω , which is a \mathfrak{g} -valued one-form, defined by having \mathcal{H} as its kernel and satisfying

$\omega(\sigma(A)|_q) = A$. It is simple to verify that if Υ is a vertical vector field along an \mathcal{H} -horizontal curve $\tilde{\gamma}$ in Q , and $\pi(\tilde{\gamma}) = \gamma$, then

$$\nabla_{\dot{\gamma}} \Upsilon = 0, \quad \text{if and only if} \quad \Upsilon(t) = \sigma(A)|_{\tilde{\gamma}(t)} \quad \text{for some } A \in \mathfrak{g}.$$

Introduce the curvature two-form Ω by $\Omega(\tilde{X}, \tilde{Y}) = d\omega(\tilde{X}, \tilde{Y}) + \omega([\tilde{X}, \tilde{Y}])$. Then we can rewrite (1) as

$$\dot{\lambda} = \vec{H}|_{\lambda} - \mathfrak{v}|_{\lambda} L\Omega(\dot{\gamma}, \cdot),$$

for some constant $L \in \mathfrak{g}^*$. If M is a Riemannian manifold, and H is the corresponding Hamiltonian, then we can write Corollary 1 as

$$\nabla_{\dot{\gamma}} \dot{\gamma} = -\sharp L\Omega(\dot{\gamma}, \cdot).$$

These are the equations for a particle with gauge L under the influence of the Yang-Mills field Ω . If $G = U(1)$ or \mathbb{R} , then L represents the charge and Ω is a magnetic field. For more details, see [13] and [14, Chapter 12].

4. OPTIMAL CONTROL OF ROLLING MANIFOLDS

4.1. Definition of manifolds rolling without twisting or slipping. In this section, unless otherwise stated, M and \widehat{M} will denote connected oriented Riemannian manifolds, both of dimension n . The Riemannian metrics of the respective manifolds will be denoted by \mathbf{g} and $\widehat{\mathbf{g}}$. To avoid trivial considerations, we will always assume that $n \geq 2$.

We consider the kinematic system of M rolling on \widehat{M} . The configuration space for this motion can be described as follows. Define the fiber bundle Q over $M \times \widehat{M}$ as

$$(6) \quad Q = \left\{ \text{SO}(T_m M, T_{\widehat{m}} \widehat{M}) : m \in M, \widehat{m} \in \widehat{M} \right\}.$$

Here, $\text{SO}(V, \widehat{V})$ denotes the space of all linear, orientation preserving isometries between two oriented inner product spaces V and \widehat{V} . Clearly each fiber is diffeomorphic to $\text{SO}(n)$. An element $q : T_m M \rightarrow T_{\widehat{m}} \widehat{M}$ in Q represents a configuration where M lie tangent to \widehat{M} at the points m and \widehat{m} . Two vectors, one in each tangent space, lie adjacent if one is mapped to the other by q . A rolling is then a curve in this configuration space.

We will assume that we have high friction, giving us the constraints that we cannot slip or twist. By slipping, we mean moving one of the manifolds while remaining connected in the same point on the other. By twisting, we mean spinning in place, that is, we move in the same fiber over $M \times \widehat{M}$, changing configuration, but not connecting points. Mathematically, we can write these constraints as follows

Let $\pi : Q \rightarrow M$ and $\widehat{\pi} : Q \rightarrow \widehat{M}$ be the respective natural projections.

Definition 1. *An absolutely continuous curve $q(t)$ in Q , with $\gamma(t) = \pi(q(t))$ and $\widehat{\gamma}(t) = \widehat{\pi}(q(t))$, is a rolling without slipping or twisting, if it satisfies the following conditions for almost every t ,*

- (No slipping condition:) $q(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$,
- (No twisting condition:) Any vector field $X(t)$ along $\gamma(t)$ is parallel if and only if $q(t)X(t)$ is parallel along $\hat{\gamma}(t)$.

In what follows, we will typically drop the phrase “without twisting or slipping” and just refer to a rolling, with the constraints being implicit. For more details on these types of systems, see e.g. [15, 6, 5].

Rolling without slipping or twisting can be described as horizontal curve relative to a subbundle D of TQ of rank n . We describe this subbundle locally. On a sufficiently small neighborhood U on M , choose an orthonormal basis e_1, \dots, e_n of vector fields on U . Define \tilde{U} and $\hat{e}_1, \dots, \hat{e}_n$ similarly on \widehat{M} . We can then trivialize the Q over $U \times \widehat{U}$ by

$$\begin{aligned} Q|_{U \times \widehat{U}} &\rightarrow U \times \widehat{U} \times \text{SO}(n) \\ q \in \text{SO}(T_m M, T_{\widehat{m}} \widehat{M}) &\mapsto (m, \widehat{m}, (q_{ij})) \quad , \quad q_{ij} = \widehat{\mathbf{g}}(\hat{e}_i|_{\widehat{m}}, q(t)e_j|_m). \end{aligned}$$

Relative to this trivializations, define vector fields

$$W_{\alpha\beta} = \sum_{s=1}^n (q_{s\alpha} \partial_{q_{s\beta}} - q_{s\beta} \partial_{q_{s\alpha}}).$$

Then D on $Q|_{U \times \widehat{U}}$ is spanned by vector fields

$$(7) \quad \bar{e}_j = e_j + qe_j + \sum_{1 \leq \alpha < \beta \leq n} (\mathbf{g}(e_\alpha, \nabla_{e_j} e_\beta) - \widehat{\mathbf{g}}(qe_\alpha, \nabla_{qe_j} qe_\beta)) W_{\alpha\beta}.$$

Here, qe_j denotes the vector field $q \mapsto qe|_{\pi(q)}$.

Notice that D is an Ehresmann connection for both $\pi : Q \rightarrow M$ and $\widehat{\pi} : Q \rightarrow \widehat{M}$. We can lift the Riemannian metric \mathbf{g} on M to a metric \mathbf{h} on D , by

$$\mathbf{h}(v_1, v_2) = \mathbf{g}(\pi_* v_1, \pi_* v_2), \quad v_1, v_2 \in D.$$

In this metric, the vector fields in (7) form a local orthonormal basis. Notice that from the definition of D , the metric \mathbf{h} also satisfies $\mathbf{h}(v_1, v_2) = \widehat{\mathbf{g}}(\widehat{\pi}_* v_1, \widehat{\pi}_* v_2)$. Hence, the sub-Riemannian structure (D, \mathbf{h}) can be considered lifted either from M or from \widehat{M} .

To find a sub-Riemannian length minimizer on Q relative to \mathbf{h} with initial point q_0 and final point q_1 , is to find the rolling $q(t)$ from q_0 to q_1 such that $\gamma(t) = \pi(q(t))$ (or equivalently $\hat{\gamma}(t) = \widehat{\pi}(q(t))$) have minimal length. This problem can be quite complicated to attack, and few results exists. The first results was given for a sphere rolling on a plane [10]. It was shown that in this case, projection of geodesics to either the sphere or the plane, are curves with constant speed whose direction is a solution of the pendulum equation. Later results for more general manifolds of constant curvature are found in [11]. In addition, it was shown in [7] that if the Riemannian curvatures R and \widehat{R} of respectively M and \widehat{M} satisfy the condition that the map

$$v_1 \wedge v_2 \mapsto R(\pi_* v_1, \pi_* v_2, \pi_* \cdot, \pi_* \cdot) - \widehat{R}(\widehat{\pi}_* v_1, \widehat{\pi}_* v_2, \widehat{\pi}_* \cdot, \widehat{\pi}_* \cdot), \quad v_1, v_2 \in D,$$

from $\bigwedge^2 D$ to $\bigwedge^2 D^*$ is invertible, then we both have that D is bracket generating and there are no abnormal minimizers which are not also normal minimizers. Hence, in this case, we can focus on normal geodesics.

We would like to describe the geodesics in the general case. Unfortunately, computations in Q defined as in (6) has proved to be quite difficult. We therefore use Theorem 1 (a) to lift our problem to a space where computations can be done more easily. We start with some definitions.

4.2. Frame bundles. We make the convention that whenever we mention \mathbb{R}^n , it will always come equipped with the standard orientation and the Euclidean inner product. We will write $\mathrm{GL}(V, \widehat{V})$ for the space of all linear isomorphisms of vector spaces V to \widehat{V} . A frame at the point $m \in M$, is a map $f \in \mathrm{GL}(\mathbb{R}^n, T_m M)$. This can be considered as a basis $\{f_1, \dots, f_n\}$ of $T_m M$, by identifying f with the vectors

$$f_j := f(\underbrace{0, \dots, 0, 1, 0, \dots, 0}_{1 \text{ in the } j\text{-th place}}).$$

Conversely, any choice of basis determines a map f . If we denote by $\mathcal{F}_m(M) = \mathrm{GL}(\mathbb{R}^n, T_m M)$, we can define a frame bundle $\tilde{\tau} : \mathcal{F}(M) \rightarrow M$ as the principal $\mathrm{GL}(n)$ -bundle with fibers $\mathcal{F}_m(M)$.

For a given affine connection ∇ on M , we define a corresponding Ehresmann \mathcal{E} on $\tilde{\tau}$ consisting of all tangent vectors of curves $f(t)$ such that each $f_1(t), \dots, f_n(t)$ is parallel vector along $\tilde{\tau}f(t)$. This sub-bundle is invariant under the right action of $\mathrm{GL}(n)$.

For an oriented Riemannian manifold, define the oriented orthonormal frame bundle $\tau : F(M) \rightarrow M$ as the principle $\mathrm{SO}(n)$ -bundle, such that the fiber over m is $F_m(M) := \mathrm{SO}(\mathbb{R}^n, T_m M)$, that can be identified with the space of all positively orthonormal frames of $T_m M$.

Let us consider the case where ∇ as the Levi-Civita connection on M . Since parallel translation along a curve preserves orthogonality, we can induce a principal Ehresmann connection \mathcal{E} on $\tau : F(M) \rightarrow M$ instead of $\mathcal{F}(M)$. This Ehresmann connection again corresponds to an $\mathfrak{so}(n)$ -valued principal connection one-form $\omega = (\omega_{\alpha\beta})$ on $F(M)$ defined as in Section 3.4.

Let $\flat : TM \rightarrow T^*M$ be the vector bundle isomorphism $v \mapsto \mathbf{g}(v, \cdot)$ and write $\sharp = \flat^{-1}$. Define an \mathbb{R}^n valued one-form $\theta = (\theta_i)$ on $F(M)$, by

$$\theta_i|_f = \tau^* \flat f_j.$$

The differential of θ and ω are connected through the Cartan equations,

$$d\theta_i = - \sum_{k=1}^n \omega_{ik} \wedge \theta_k.$$

$$d\omega_{\alpha\beta} = - \sum_{k=1}^n \omega_{\alpha k} \wedge \omega_{k\beta} + \Omega_{\alpha\beta}.$$

where $\Omega = (\Omega_{\alpha\beta})$ is the $\mathfrak{so}(n)$ -valued curvature two-form. In this case, this is connected to the Riemannian curvature by

$$\Omega(f)(X, Y) = (R(X, Y, f_\beta, f_\alpha))_{\alpha, \beta}.$$

$$R(X, Y, Z, W) = \mathbf{g}((\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]})Z, W).$$

We can rewrite the Cartan equations in terms of vector fields rather than forms. For any $A = (A_{\alpha\beta}) \in \mathfrak{so}(n)$, let $\sigma(A)$ be defined as in (5). These satisfy $\theta_i(\sigma(A)) = 0$ and $\omega_{\alpha\beta}(\sigma(A)) = A_{\alpha\beta}$. We also have the vector fields X_i with values in \mathcal{E} , defined by $X_i|_f = h_f f_i$, where the horizontal lift is with respect to \mathcal{E} . These satisfy $\theta_i(X_j) = \delta_{i,j}$ and $\omega_{\alpha\beta}(X_j) = 0$. Using this information along with the Cartan equations, we obtain

$$[X_i, X_j] = -\sigma(\Omega(X_i, X_j)).$$

$$[X_j, \sigma(A)] = -\sum_{s=1}^n A_{sj} X_s.$$

$$[\sigma(A), \sigma(B)] = \sigma([A, B]).$$

4.3. Lifting of the rolling problem. To simplify our computations, we want to lift our problem to $F(M) \times F(\widehat{M})$. Define a principal $\mathrm{SO}(n)$ -bundle by

$$\begin{aligned} \mathbf{p}: F(M) \times F(\widehat{M}) &\rightarrow Q \\ (f, \widehat{f}) &\mapsto \widehat{f} \circ f^{-1}. \end{aligned}$$

We will use the following result.

Lemma 2. [7, Cor 1] *Let $q(t)$ be a curve in Q which projects to the curve $(\gamma(t), \widehat{\gamma}(t))$ in $M \times \widehat{M}$. Then $q(t)$ is a rolling without twisting or slipping if and only if it is a projection of a curve $(f(t), \widehat{f}(t))$ in \widetilde{Q} such that*

- (a) $f_1(t), \dots, f_n(t)$ are parallel along $\gamma(t)$,
- (b) $\widehat{f}_1(t), \dots, \widehat{f}_n(t)$ are parallel along $\widehat{\gamma}(t)$,
- (c) $f(t)^{-1}(\dot{\gamma}(t)) = \widehat{f}(t)^{-1}(\dot{\widehat{\gamma}}(t))$,

Such curves $(f(t), \widehat{f}(t))$ may be considered as horizontal lifts of the curve $q(t)$ with respect to an Ehresmann connection \mathcal{H} on \mathbf{p} . We will describe this connection.

Let $\theta, \omega, \Omega, X_j$ and $\sigma(A)$ be defined on $F(M)$ as in Section 4.2, and define $\widehat{\theta}, \widehat{\omega}, \widehat{\Omega}, \widehat{X}_j$ and $\widehat{\sigma}(A)$ similarly on $F(\widehat{M})$. Then

$$\mathcal{V} = \ker \mathbf{p}_* = \{\sigma(A) + \widehat{\sigma}(A) : A \in \mathfrak{so}(n)\}.$$

Pick an Ehresmann connection

$$\mathcal{H} = \mathrm{span}\{X_j, \widehat{X}_j, \sigma(A) - \widehat{\sigma}(A) : j = 1, \dots, n, A \in \mathfrak{so}(n)\}.$$

Use horizontal lifts with respect to \mathcal{H} of elements in D to define a distribution \widetilde{D} . Then \widetilde{D} -horizontal curves in $F(M) \times F(\widehat{M})$ are exactly the curves in Lemma 2. A basis for \widetilde{D} is given by $X_j + \widehat{X}_j$. Lift the metric \mathbf{h} to a metric $\widetilde{\mathbf{h}}$ on \widetilde{D} . Then $X_j + \widehat{X}_j$ form an orthonormal basis. By Corollary 2, we

can now find optimal solutions of the rolling problem by looking for normal geodesics of the sub-Riemannian manifold $(F(M) \times F(\widehat{M}), \widetilde{D}, \widehat{\mathbf{h}})$ which are projections of integral curves in $\text{Ann}(\mathcal{V})$ of the sub-Riemannian Hamiltonian with respect to $\widehat{\mathbf{h}}$.

4.4. Normal geodesics and optimal solutions. We will use Theorem 1(b) to describe the normal geodesics. In order to present our results, let us introduce the $\mathfrak{so}(n)$ -valued two-form on $F(M)$ defined by

$$\Omega^{S^n}(\sigma(A), \cdot) = 0, \quad \Omega^{S^n}(X_r, X_s) = -(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj})_{i,j}.$$

This notation is inspired by the fact that this form coincides with the curvature form when $M = S^n$. We will also identify $\mathfrak{so}(n)$ with $\mathfrak{so}(n)^*$ through the inner product

$$(8) \quad \langle A, B \rangle = \text{tr}(A^\top B).$$

With slight abuse of notation, we also will use \flat for this identification $\flat : \mathfrak{so}(n) \rightarrow \mathfrak{so}(n)^*$, with inverse \sharp .

Theorem 4. *Let $q(t)$ be a horizontal curve in (Q, D, \mathbf{h}) with a projection $\gamma(t)$ in M . Then $q(t)$ is a normal geodesic if and only if there is vector fields $V(t)$ along $\gamma(t)$ and a curve $L(t)$ in $\mathfrak{so}(n)^*$ satisfying*

$$\begin{aligned} \nabla_{\dot{\gamma}} \dot{\gamma} &= -\sharp L \left(\Omega(\dot{\gamma}, \cdot) - \widehat{\Omega}(q\dot{\gamma}, q\cdot) \right) \\ \sharp \dot{L} &= -\frac{1}{2} \Omega^{S^n}(\dot{\gamma}, V), \quad \nabla_{\dot{\gamma}} V = -\sharp L \widehat{\Omega}(q\dot{\gamma}, q\cdot). \end{aligned}$$

Remark 5. At first glance, it may seem striking that the curvature tensor of S^n appears in the geodesic equations in Theorem 4. However, its appearance is just the result of our choice of inner product in (8) to give $\mathfrak{so}(n)^*$ convenient coordinates.

There is also an apparent asymmetry between M and \widehat{M} , since $\nabla_{\dot{\gamma}} V$ only depends on the curvature of \widehat{M} . This is of course an illusion. Define $\widetilde{V}(t) = V(t) + \dot{\gamma}$. Then it is still true that $\sharp \dot{L} = -\frac{1}{2} \Omega^{S^n}(\dot{\gamma}, \widetilde{V})$, but now we have $\nabla_{\dot{\gamma}} \widetilde{V} = -\sharp L \Omega(\dot{\gamma}, \cdot)$.

Proof. Let $(f(t), \widehat{f}(t))$ be some \widetilde{D} -horizontal lift of $\gamma(t)$ and let $q(t) = \widehat{f}(t) \circ f(t)^{-1}$. We will use Corollary 1 to find the geodesics, recalling that \widetilde{D} is an Ehresmann connection on $F(M) \times F(\widehat{M}) \rightarrow M$.

We start by finding the curvature $\widetilde{\mathcal{R}}$ of \widetilde{D} . Since elements $X_i + \widehat{X}_i$ form a basis, the curvature is $\widetilde{\mathcal{R}}(X_i + \widehat{X}_i, X_j + \widehat{X}_j) = \sigma(\Omega(X_i, X_j)) + \widehat{\sigma}(\widehat{\Omega}(X_i, X_j))$.

Next, we determine the equations for parallel transport. Write $\dot{\gamma}(t) = \sum_{i=1}^n \dot{\gamma}_i(t) e_i|_{\gamma(t)}$ for some local orthonormal basis e_1, \dots, e_n . Introduce local coordinates $f_j = \sum_{i=1}^n f_{ij} e_i$. Using these coordinates

$$\sigma(A) = \sum_{i,j,s=1}^n f_{is} A_{sj} \partial_{f_{ij}}.$$

Since the \tilde{D} -horizontal lift of e_i is $he_i = \sum_{s=1}^n f_{is}(X_s + \hat{X}_s)$, we have

$$\nabla_{e_i} \hat{X}_j = -\hat{\sigma}(\hat{\Omega}(he_i, \hat{X}_j)).$$

$$\nabla_{e_i} \sigma(A) = -\nabla_{e_i} \hat{\sigma}(A) = \sum_{r,s=1}^n f_{ir} A_{sr} \hat{X}_s.$$

If $v(t)$ is a curve in $\text{Ann}(\mathcal{V}) \cap \text{Ann}(\tilde{D})$, then it is on the form $\sum_{j=1}^n v_j(t)(\theta_j - \hat{\theta}_j) + \frac{1}{2} \sum_{i,j=1}^n L_{ij}(t)(\omega_{ij} - \hat{\omega}_{ij})$ where $\sharp L = (L_{ij})$ is a curve in $\mathfrak{so}(n)$. In order for v to be parallel, it must satisfy

$$\dot{v}_j = -\partial_t \alpha(\hat{X}_j) = -\alpha(\nabla_{\dot{\gamma}} \hat{X}_j) = -L\hat{\Omega}(h\dot{\gamma}, \hat{X}_j)$$

$$\begin{aligned} \langle \sharp \dot{L}, A \rangle &= \partial_t \alpha(\sigma(A)) = \alpha(\nabla_{\dot{\gamma}} \sigma(A)) = - \sum_{i,r,s=1}^n \dot{\gamma}_i f_{ir} A_{sr} v_s \\ &= -\frac{1}{2} \left\langle \left(\sum_{i=1}^n (\dot{\gamma}_i f_{ir} v_s - \dot{\gamma}_i f_{is} v_r) \right)_{rs}, A \right\rangle = -\frac{1}{2} \langle \Omega^{S^2}(\dot{\gamma}, V), A \rangle \end{aligned}$$

where V is the vector field $V(t) = \sum_{i=1}^n v_i(t) f_i(t)$.

We insert these results into formula Corollary 1, and we find

$$\begin{aligned} \nabla_{\dot{\gamma}} \dot{\gamma} &= -\sharp v(t) \mathcal{R}(\dot{\gamma}, \cdot) \\ &= -\sharp \left(\sum_{j=1}^n v_j(t)(\theta_j - \hat{\theta}_j) + \frac{1}{2} \sum_{i,j=1}^n L_{ij}(t)(\omega_{ij} - \hat{\omega}_{ij}) \right) \left(\sigma(\Omega(\dot{\gamma}, \cdot)) + \hat{\sigma}(\hat{\Omega}(q\dot{\gamma}, \cdot)) \right) \\ &= -\sharp L \left(\Omega(\dot{\gamma}, \cdot) - \hat{\Omega}(q\dot{\gamma}, \cdot) \right) \end{aligned}$$

where $\sharp L = -\frac{1}{2} \langle \Omega^{S^2}(\dot{\gamma}, V), A \rangle$ and $\nabla_{\dot{\gamma}} V = -\frac{1}{2} \hat{\Omega}(q\dot{\gamma}, q \cdot)$. \square

4.5. The two dimensional case. Let us consider the case when the Riemannian manifolds M and \widehat{M} are two-dimensional. Let $f_1(t)$ and $f_2(t)$ be a basis of orthogonal parallel vector fields along a curve $\gamma(t)$ in M . Write

$$(9) \quad \dot{\gamma} = a(\cos \theta f_1 + \sin \theta f_2).$$

Use respectively Ω and $\hat{\Omega}$ for the Gaussian curvatures of M and \widehat{M} . We will make the addition assumption that for any pair of points $(m, \hat{m}) \in M \times \widehat{M}$ we have that $\Omega|_m - \hat{\Omega}|_{\hat{m}}$ never vanishes. The rolling distribution is bracket generating if and only if this condition holds [4, 1]. Under this assumption, we have the following result.

Proposition 1. *Let $q(t)$ be an absolutely continuous curve in Q , with projection $\gamma(t)$. Write $\Omega(t) = \Omega|_{\gamma(t)}$ and $\hat{\Omega}(t) = \hat{\Omega}|_{q(t)\gamma(t)}$, and introduce the notation $\rho(t) = \frac{1}{\Omega(t) - \hat{\Omega}(t)}$. Represent γ as in (9). Then $q(t)$ is a normal geodesic if and only if a is constant, and θ is a solution to the equation*

$$(10) \quad \ddot{\theta} + \frac{\dot{\rho}}{\rho} \dot{\theta} = \frac{A}{\rho} \sin(\theta - \phi_0) - \frac{a^2}{4\rho} F,$$

where $F(t) = \int_0^t \rho(s)^2 \sin(\theta(t) - \theta(s))(\Omega(s)\dot{\hat{\Omega}}(s) - \dot{\Omega}(s)\hat{\Omega}(s))ds$ and A and ϕ_0 are arbitrary constants.

Remark 6. If $\Omega(t)\dot{\hat{\Omega}}(t) = \dot{\Omega}(t)\hat{\Omega}(t)$, i.e. if one of the Gaussian curvatures is a constant times the other, the geodesic equations reduces to the equation a pendulum. More specifically, when $F(t) = 0$, (10) is the equation for the pendulum whose length varies by $\rho(t)$ under the influence of the force of gravity with magnitude A and direction θ_0 . See Section 5.3 for details.

In particular, this happens if both of the manifolds have constant Gaussian curvature or if one of the manifolds is flat.

Proof. It is simple to verify from Theorem 4 that $|\dot{\gamma}|$ is always a first integral. In the notation of the same theorem, let $V = b_1(\cos \theta f_1 + \sin \theta f_2) + b_2(-\sin \theta f_1 + \cos \theta f_2)$. Also write $\sharp L$ and the curvature forms as

$$\frac{1}{2} \begin{pmatrix} 0 & L \\ -L & 0 \end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}, \quad \text{and} \quad \frac{1}{2} \begin{pmatrix} 0 & \hat{\Omega} \\ -\hat{\Omega} & 0 \end{pmatrix}.$$

We can rewrite equations in Theorem 4 in these variables.

$$\dot{\theta} = -\frac{1}{2}L(\Omega - \hat{\Omega}),$$

$$\dot{L} = \frac{1}{2}ab_2, \quad \dot{b}_1 = \dot{\theta}b_2, \quad \dot{b}_2 = -\dot{\theta}b_1 - \frac{1}{2}aL\hat{\Omega}.$$

This can be reduced to three equations by the identity $L = -2\frac{\dot{\theta}}{\Omega - \hat{\Omega}} = -2\rho\dot{\theta}$, which give us the formulation

$$(11) \quad \dot{b}_1 = \dot{\theta}b_2 \quad \dot{b}_2 = -\dot{\theta} \left(b_1 + a \frac{\hat{\Omega}}{\Omega - \hat{\Omega}} \right).$$

$$-2\rho\ddot{\theta} - 2\dot{\rho}\dot{\theta} = \frac{1}{2}ab_2.$$

By using a change of variable and a standard variation of parameters argument, we can solve (11). We find that

$$b_2(t) = A \sin(\theta(t) - \phi_0) + a \int_0^t \sin(\theta(t) - \theta(s)) \frac{\Omega(s)\dot{\hat{\Omega}}(s) - \dot{\Omega}(s)\hat{\Omega}(s)}{(\Omega(s) - \hat{\Omega}(s))^2} ds,$$

which completes the proof. \square

4.6. Rolling on \mathbb{R}^n . Assume that $\widehat{M} = \mathbb{R}^n$ with the Euclidean metric and standard orientation. Then we can identify Q with $\mathbb{R}^n \times F(M)$, by associating $q : T_m M \rightarrow T_y \mathbb{R}^n$ with the pair (y, q^{-1}) , where we have used the canonical identification of $T_y \mathbb{R}^n$ with \mathbb{R}^n . We then have the following reformulation of Theorem 4.

Corollary 3. *Let $q(t)$ be a rolling of M on \mathbb{R}^n .*

- (a) Let $\gamma(t)$ be its projection to M . Then $q(t)$ is a normal geodesic if and only if

$$\nabla_{\dot{\gamma}}\dot{\gamma} = -\sharp L_0\Omega(\dot{\gamma}, \cdot) + \sharp R(V, W, \dot{\gamma}, \cdot),$$

where $\nabla_{\dot{\gamma}}V = 0$ and $\nabla_{\dot{\gamma}}W = \dot{\gamma}$, and L_0 is a constant element in $\mathfrak{so}(n)^*$.

- (b) Identify $q(t)$ with the curve $(y(t), f(t))$ in $\mathbb{R}^n \times F(M)$. Then this is a normal geodesic if and only if

$$\ddot{y} = -\sharp L_0\Omega(f(t)\dot{y}(t), \cdot) + \sharp R(f(t)v, f(t)(y(t) - y(0)), f(t)\dot{y}(t), \cdot),$$

for some $v \in \mathbb{R}^n$ and $L_0 \in \mathfrak{so}(n)^*$.

Proof. Here we have used that V becomes parallel when $\widehat{\Omega} = 0$. Also, since V , now is parallel, if W is any other vector field along γ , then $\partial_t \Omega^{S^n}(W, V) = \Omega^{S^n}(\nabla_{\dot{\gamma}}W, V)$. The result then follows \square

Let us do a physical interpretation of these equations in the two dimensional case. The curvature form Ω on M can be considered as an \mathbb{R} -valued two form, and we write this as $\Omega = B \text{ vol}$, where vol is the Riemannian volume form induce by the metric and orientation on M . Ω represents a magnetic field with magnitude B , always normal to the manifold.

Let $\gamma(t)$ be the trajectory of a charged object moving in M . The initial charge is L_0 , however this charge changes according to a parallel vector field V . Let X be a vector field on M extending V locally. Then the charge at time t is given by $L_0 + \int_{\gamma|_{[0,t]}} \star bX$, where \star is the Hodge star operator. In other words, according to the positive orientation, movement in a direction rotated 90° from the direction of X will increase the charge, while movement in the opposite directions decreases the charge.

5. PROOFS

5.1. Proof of Theorem 1. Before we are ready to present the proof in Section 5.1.3, we need to introduce some concepts, such as lifted vertical and horizontal bundles and the π^2 -symplectic lift of vectors. These concepts and some necessary lemmas are described in Sections 5.1.1 and 5.1.2, with a representation in local coordinates included in Section 5.1.4.

5.1.1. Lifted horizontal and vertical bundle. Let $\pi^2 : T^*Q \rightarrow T^*M$ be the submersion defined in Section 2.3, relative to some Ehresmann connection \mathcal{H} on $\pi : Q \rightarrow M$. From the first mentioned submersion, we have vertical space $\mathcal{V}^2 = \ker \pi^2_*$. We will show that this subbundle of TT^*Q is symplectic with respect to the canonical symplectic form on T^*Q and we can use this to define an Ehresmann connection \mathcal{H}^2 on π^2 .

Elements in the cotangent bundles of Q and M are denoted by respectively \tilde{p} and p . We will use ϑ for the Liouville one-form on T^*M defined by

$\vartheta(w) = p(\Pi^{T^*M} w)$, $w \in T_p T^*M$ and $\varsigma = -d\vartheta$ for the canonical symplectic form. Use I for the bundle morphism $I : TT^*M \rightarrow T^*T^*M$, $I(w) = \varsigma(w, \cdot)$. Define $\tilde{\vartheta}, \tilde{\varsigma}$ and \tilde{I} similarly on Q . Relative these structures, we will introduce a notation for what we will call *the π^2 -symplectic lift*. For any $w \in T_p T^*M$ and $\tilde{p} \in (\pi^2)^{-1}(p)$, we will use $\ell_{\tilde{p}} w$ for the element

$$\ell_{\tilde{p}} w = \tilde{I}^{-1} \pi^{2*}|_{\tilde{p}} I(w).$$

Here, by abusing notation, we have used $\pi^{2*}|_{\tilde{p}}$ for the map $T_p^* TM \rightarrow T_{\tilde{p}}^* TQ$, $\alpha_0 \mapsto \alpha_0(\pi^{2*}|_{\tilde{p}} \cdot)$. In other words, $\ell_{\tilde{p}} w$ is the unique element in $T_{\tilde{p}}^* TQ$ satisfying

$$\tilde{\varsigma}(\ell_{\tilde{p}} w, \tilde{u}) = \varsigma(w, \pi^{2*} \tilde{u}), \quad \text{for any } \tilde{u} \in T_{\tilde{p}} TQ.$$

We use this notation in the following lemma.

Lemma 3. *Let \mathcal{H}^2 be the symplectic complement of \mathcal{V}^2 , i.e. $\mathcal{H}^2 = \tilde{I}^{-1} \text{Ann}(\mathcal{V}^2)$.*

- (a) *For any $\tilde{p} \in T^*Q$, the fiber of \mathcal{H}^2 is given by $\mathcal{H}_{\tilde{p}}^2 = \{\ell_{\tilde{p}} w : w \in T_{\pi^2(\tilde{p})} T^*M\}$.*
- (b) *For any $\tilde{p} \in T^*Q$ and $w \in T_{\pi^2(\tilde{p})} T^*M$, we have*

$$\pi_* \Pi^{T^*Q} \ell_{\tilde{p}} w = \Pi^{T^*M} w.$$

- (c) *\mathcal{V}^2 and \mathcal{H}^2 are symplectic, i.e. $\mathcal{H}^2 \oplus \mathcal{V}^2 = TT^*Q$. Furthermore,*

$$\Pi^{T^*Q} \mathcal{V}^2 = \mathcal{V} \quad \text{and} \quad \Pi^{T^*Q} \mathcal{H}^2 = \mathcal{H}.$$

Proof. (a) This is an easy consequence of the fact that for any $\tilde{p} \in T^*Q$, the map $T_{\pi^2(\tilde{p})}^* T^*M \rightarrow \text{Ann}(\mathcal{V}^2)_{\tilde{p}}$, $\alpha_0 \mapsto \pi^{2*} \alpha_0$ is a linear isomorphism.

- (b) For any two elements $p, \alpha_0 \in T_m^* M$, it is simple to verify that

$$\varsigma(\text{vl}_p \alpha_0, w) = \alpha_0(\Pi^{T^*M} w), \quad \text{for any } w \in T_p T^*M.$$

Observe that for any one-form α on T^*M , by definition $\pi^{2*} \text{vl} \pi^* \alpha = \text{vl} \alpha$. Hence, for any vector field W on T^*M , we have

$$\alpha(\Pi^{T^*M} W) = -\varsigma(W, \text{vl} \alpha) = -\tilde{\varsigma}(\ell W, \text{vl} \pi^* \alpha) = \alpha(\pi_* \Pi^{T^*Q} \ell W).$$

Here, the vector field ℓW on T^*Q is defined in the obvious way. Since α and W where arbitrary, the result follows.

- (c) Clearly, $\Pi^{T^*Q} \mathcal{V}^2 = \Pi^{T^*Q} \ker \pi^{2*} \subseteq \ker \pi_* = \mathcal{V}$. This map has to be surjective since it is a right inverse to the injective map $\mathbf{0}^{T^*Q} : TQ \rightarrow TT^*Q$ which maps \mathcal{V} into \mathcal{V}^2 .

To prove the analogous result for \mathcal{H}^2 , observe that for any $\tilde{p} \in T_q^* Q$ and $\tilde{\alpha}_0 \in \text{Ann}(\mathcal{H})_q$, we have $\text{vl}_{\tilde{p}} \tilde{\alpha}_0 \in \mathcal{V}^2$. By definition $\tilde{\vartheta}(\text{vl}_{\tilde{p}} \tilde{\alpha}_0, \tilde{w}) = \tilde{\alpha}_0(\Pi^{T^*Q} \tilde{w}) = 0$ for any $\tilde{w} \in \mathcal{H}^2$. Hence, $\Pi^{T^*Q} \mathcal{H}^2 \subseteq \mathcal{H}$ and this map is surjective by (b).

Finally, assume that $\tilde{w} \in (\mathcal{H}^2 \cap \mathcal{V}^2)|_{\tilde{p}}$. From our previous results, this intersection is contained in the kernel of $\Pi^{T^*Q} \cdot$. Hence, $\tilde{w} = \text{vl}_{\tilde{p}} \tilde{\alpha}_0$ for some $\tilde{\alpha}_0 \in T^*Q$ which is in the same fiber as \tilde{p} . However, since $\tilde{\varsigma}(\text{vl}_{\tilde{p}} \alpha_0, \cdot)$ must annihilate both \mathcal{H}^2 and \mathcal{V}^2 and since

$\mathcal{H}^2 \cup \mathcal{V}^2$ is mapped surjectively onto TM , the identity $\zeta(\text{vl}_{\tilde{p}} \tilde{\alpha}_0, \tilde{u}) = \tilde{\alpha}_0(\Pi^{T^*Q}_* \tilde{u})$ implies that $\tilde{\alpha}_0 = 0$. This shows that \mathcal{H}^2 and \mathcal{V}^2 are transversal. \square

5.1.2. *Horizontal lifts with respect to \mathcal{H}^2 .* A consequence of Lemma 3 (c), is that \mathcal{H}^2 is an Ehresmann connection on $\pi^2 : T^*Q \rightarrow T^*M$. Hence, we have two different ways of lifting elements in TT^*M to \mathcal{H}^2 , namely the π^2 -symplectic lift and the horizontal lift. We will denote the latter by $h_{\tilde{p}}^2 w$. The following lemma describes the relation of these two notions.

Lemma 4. *If $\tilde{p} \in T_q^*Q$, the following relations hold.*

(a)

$$\Pi^{T^*Q}_* h_{\tilde{p}}^2 w = \Pi^{T^*Q}_* \ell_{\tilde{p}} w = h_q \left(\Pi^{T^*M}_* w \right).$$

(b)

$$(12) \quad h_{\tilde{p}}^2 w = \ell_{\tilde{p}} w + \text{vl}_{\tilde{p}} \tilde{p} \tilde{\mathcal{R}}(h_q \Pi^{T^*M}_* w, \cdot).$$

Here, $\tilde{p} \mathcal{R}(h_q \Pi^{T^*M}_* w, \cdot) \in T_q^*Q$ is the covector $\tilde{v} \mapsto \tilde{p} \tilde{\mathcal{R}}(h_q \Pi^{T^*M}_* w, v)$. As a consequence

$$\pi^2_* \ell_{\tilde{p}} w = w - \text{vl}_p \tilde{p} \mathcal{R}(\Pi^{T^*M}_* w, \cdot) \quad p = \pi^2(\tilde{p}).$$

Remark 7. Written as a commutative diagram, let \tilde{p} in T^*Q satisfy

$$\begin{array}{ccc} \tilde{p} & \xrightarrow{\pi^2} & p \\ \Pi^{T^*Q}_* \downarrow & & \downarrow \Pi^{T^*M} \\ q & \xrightarrow{\pi} & m \end{array}.$$

Lemma 4 then states that for any that for any $w \in T_p T^*M$ with $v = \Pi^{T^*M}_* w$, we have the relations

$$\begin{array}{ccc} h_{\tilde{p}}^2 w & \xrightarrow{\pi^2_*} & w \\ \Pi^{T^*Q}_* \downarrow & & \downarrow \Pi^{T^*M}_* \\ h_q v & \xrightarrow{\pi_*} & v \end{array} \quad \text{and} \quad \begin{array}{ccc} \ell_{\tilde{p}} w & \xrightarrow{\pi^2_*} & w - \text{vl}_p \tilde{p} \mathcal{R}(v, \cdot) \\ \Pi^{T^*Q}_* \downarrow & & \downarrow \Pi^{T^*M}_* \\ h_q v & \xrightarrow{\pi_*} & v \end{array}$$

Observe that $\text{vl}_p \tilde{p} \mathcal{R}(v, \cdot)$ only depends on the projection of \tilde{p} to $\text{Ann}(\mathcal{H})$. Hence, it is simple to see that $h_{\tilde{p}}^2 w = \ell_{\tilde{p}} w$ whenever \tilde{p} is inn $\text{Ann}(\mathcal{V})$.

Proof of Lemma 4. (a) This is immediate from the definition of horizontal lift and Lemma 3 (b) and (c).

(b) Let us split the Liouville form on T^*Q into two parts $\tilde{\vartheta} = \vartheta^{\mathcal{H}} + \vartheta^{\mathcal{V}}$ where

$$\vartheta^{\mathcal{H}}(w) = \tilde{p}(\text{pr}_{\mathcal{H}} \Pi^{T^*Q}_* \tilde{w}), \quad \vartheta^{\mathcal{V}}(w) = \tilde{p}(\text{pr}_{\mathcal{V}} \Pi^{T^*Q}_* \tilde{w}), \quad \tilde{w} \in T_{\tilde{p}} T^*Q.$$

We observe that for any $\tilde{w} \in T_{\tilde{p}}T^*Q$,

$$\begin{aligned}\vartheta^{\mathcal{H}}(\tilde{w}) &= \tilde{p}(\text{pr}_{\mathcal{H}} \Pi^{T^*Q} \tilde{w}) = \pi^2(\tilde{p})(\pi_* \Pi^{T^*Q} \tilde{w}) \\ &= \pi^2(\tilde{p})(\Pi^{T^*M} \pi^2_* \tilde{w}) = \pi^{2*}(\vartheta)(\tilde{w})\end{aligned}$$

which in turn imply $\tilde{\zeta} = \pi^{2*}\zeta - d\vartheta^{\mathcal{V}}$.

Since \mathcal{H}^2 is symplectic, $h_p^2 w$ is completely determined by the values of $\tilde{I}(h_p^2 w)$ on vectors $h_p^2 u$, where $u \in T_{\pi^2(\tilde{p})}TM$. Define $\tilde{\mathcal{R}}^2$ as the curvature of \mathcal{H}^2 . From (a), we know that $\Pi^{\tilde{T}^*Q} \tilde{\mathcal{R}}^2(h_p^2 w, h_p^2 u) = \tilde{\mathcal{R}}(h_q \Pi^{T^*M} w, h_q \Pi^{T^*M} u)$. We calculate

$$\begin{aligned}\tilde{\zeta}(h_p^2 w, h_p^2 u) &= \varsigma(w, u) - d\vartheta^{\mathcal{V}}(h_p^2 w, h_p^2 u) \\ &= \varsigma(w, u) + \vartheta(\tilde{\mathcal{R}}^2(h_p^2 w, h_p^2 u)) \\ &= \varsigma(w, u) + \tilde{p} \tilde{\mathcal{R}}(h_q \Pi^{T^*M} w, h_q \Pi^{T^*M} u) \\ &= \tilde{\zeta}(\ell_{\tilde{p}} w, h_p^2 u) + \tilde{\zeta}(\text{vl}_{\tilde{p}} \tilde{p} \tilde{\mathcal{R}}(h_q \Pi^{T^*M} w, \cdot), h_p^2 u)\end{aligned}$$

which shows (12) □

We will state one of the observations made in the proof of Lemma 4 as a separate result, since we will need it later.

Lemma 5. *If $\vartheta^{\mathcal{V}}|_{\tilde{p}} = \tilde{p}(\text{pr}_{\mathcal{V}} \Pi^{T^*Q} \cdot)$ is a form on T^*Q , then*

$$\tilde{\zeta} = \pi^{2*}\zeta - d\vartheta^{\mathcal{V}}.$$

Lemma 6. *Let $\lambda(t)$ be a curve in T^*M with $\Pi^{T^*M}(\lambda(t)) = \gamma(t)$. Let $\tilde{\lambda}(t)$ be an \mathcal{H}^2 -horizontal lift of $\lambda(t)$ with $\Pi^{T^*Q}(\tilde{\lambda}(t)) = \tilde{\gamma}(t)$. Then $\tilde{\lambda}(t) = \lambda^{\mathcal{H}}(t) + \lambda^{\mathcal{V}}(t)$, where $\lambda^{\mathcal{H}}$ is the curve in $\text{Ann}(\mathcal{V})$ determined by*

$$(13) \quad \lambda^{\mathcal{H}}(t)(\tilde{v}) = \lambda(t)(\pi_* \tilde{v}) \text{ for any } \tilde{v} \in T_{\tilde{\gamma}(t)}Q,$$

and $\lambda^{\mathcal{V}}$ is a curve in $\text{Ann}(\mathcal{H})$ satisfying

$$\nabla_{\dot{\gamma}(t)} \lambda^{\mathcal{V}}(t) = 0.$$

Proof. First observe that since π^2 commutes with addition on respectively T^*Q and T^*M . Hence, both \mathcal{V}^2 and \mathcal{H}^2 are preserved under addition. In particular, the sum of two \mathcal{H}^2 -horizontal curves are again \mathcal{H}^2 .

Let $\lambda^{\mathcal{H}}(t)$ be defined as in (13). We will show that this is an \mathcal{H}^2 -horizontal lift. Clearly $\pi^2(\lambda^{\mathcal{H}}(t)) = \lambda(t)$, so all we need to show is that $\lambda^{\mathcal{H}}(t)$ is \mathcal{H}^2 -horizontal. By using the fact that we, at least for short time can write $\lambda^{\mathcal{H}}$ as $t \mapsto \pi^* \alpha|_{\tilde{\gamma}(t)}$ for some one-form α on M , it is simple to verify that $\varsigma(\dot{\lambda}^{\mathcal{H}}, \cdot)$ vanishes on \mathcal{V}^2 .

Using the conclusions from the previous two paragraphs, we obtain that $\lambda^{\mathcal{V}}(t) = \tilde{\lambda}(t) - \lambda^{\mathcal{H}}(t)$ is an \mathcal{H}^2 -horizontal lift of $\mathbf{0}^{T^*M}|_{\gamma(t)}$. Let $\tilde{\alpha}$ be a section of $\text{Ann}(\mathcal{H})$, which, at least for short time, satisfy $\tilde{\alpha}|_{\tilde{\gamma}(t)} = \tilde{\lambda}(t)$. Let Υ be any

section of \mathcal{V} , and let \widetilde{W} be a section of \mathcal{V}^2 which projects to Υ . Inserting this into the symplectic form on T^*Q , we have

$$\varsigma(\dot{\lambda}^\vee, \widetilde{W}) = \partial_t \alpha(\Upsilon)|_{\tilde{\gamma}(t)} - \widetilde{W} \alpha(\dot{\tilde{\gamma}}(t)) - \alpha(\nabla_{\dot{\tilde{\gamma}}(t)} \Upsilon) = (\nabla_{\dot{\tilde{\gamma}}(t)} \alpha)(\Upsilon) = 0.$$

Here we have used that $\tilde{\gamma}$ is \mathcal{H} -horizontal. \square

5.1.3. *Proof of Theorem 1.* Let H be any smooth function on T^*M and let $\tilde{H} = H \circ \pi^2$. By definition, it is clear that $\ell \tilde{H}$ is the Hamiltonian vector field of \tilde{H} .

To prove (a), assume that $\lambda(t)$ is a curve in T^*M along $\gamma(t)$. Let $\tilde{\gamma}$ be a horizontal lift and let $\lambda^{\mathcal{H}}$ be defined as in (13). By Lemma 4 (b), we know that $\ell \tilde{H}$ and $h^2 \tilde{H}$ coincides on $\text{Ann}(\mathcal{V})$. The result then follows from Lemma 6.

For the proof of (b), introduce the map $\Phi^\vee : T^*Q \rightarrow T^*Q$ defined by $\Phi^\vee(\tilde{p}) = \tilde{p} \text{pr}_\mathcal{V}$. Obviously

$$\Phi^\vee_*(h_p^2 w - \ell_{\tilde{p}} w) = 0,$$

from Lemma 4 (b), since $\tilde{p} \tilde{\mathcal{R}}(h_q \Pi^{T^*M}_* w, \cdot)$ vanishes on vertical vectors. As such, if $\tilde{\lambda}$ is an integral curve of \tilde{H} with $\Phi^\vee(\tilde{\lambda}) = v(t)$ and $\pi^2(\tilde{\lambda}) = \lambda$, then clearly $\nabla_{\dot{\tilde{\gamma}}} v = 0$ by Lemma 6, where γ is the projection of λ to M . We again use Lemma 4 (b), for the conclusion

$$\dot{\lambda} = \pi^2_* \ell \tilde{H}|_{\tilde{\lambda}} = \tilde{H}|_{\lambda} - \text{vl}_\lambda \tilde{\lambda} \mathcal{R}(\dot{\gamma}, \cdot) = \tilde{H}|_{\lambda} - \text{vl}_\lambda v \mathcal{R}(\dot{\gamma}, \cdot)$$

Finally, we know that $\tilde{\gamma} = \Pi^{T^*Q}(\tilde{\lambda})$ is an \mathcal{H} -horizontal lift of $\gamma = \Pi^{T^*M}(\lambda)$ from Lemma 4 (a).

5.1.4. *Representation in local coordinates.* Recall that M is n dimensional, while Q has dimension $n + \nu$. Indices running from 1 to n will be in latin letters i, j, k , while indices running from 1 to ν , will be in greek letters κ, λ, μ .

Let (x, U) be a local coordinate system on M , and let \tilde{U} be a neighborhood of $\pi^{-1}(U)$ such that \mathcal{V} trivialize over this neighborhood. Let $\Upsilon_1, \dots, \Upsilon_\nu$ be vector fields on \tilde{U} forming a basis for \mathcal{V} . We give T^*U coordinates $p = \sum_{i=1}^n p_i dx_i$ and $T^*\tilde{U}$ coordinates $\tilde{p} = \sum_{i=1}^n a_i \pi^* dx_i + \sum_{\kappa=1}^\nu b_\kappa \Upsilon_\kappa^*$. Here, Υ_κ^* is defined by formula

$$\Upsilon_\kappa^*(h \partial_{x_i}) = 0, \quad \Upsilon_\kappa^*(\Upsilon_\mu) = \delta_{\kappa\mu}.$$

Relative to these coordinates,

$$\pi^2 : \sum_{i=1}^n a_i \pi^* dx_i|_q + \sum_{\kappa=1}^\nu b_\kappa \Upsilon_\kappa^*|_q \mapsto \sum_{i=1}^n a_i dx_i|_{\pi(m)}.$$

The bundle \mathcal{V}^2 is consequently spanned by Υ_κ and ∂_{b_κ} . Write

$$\mathcal{R}_{ij}^\kappa = \Upsilon_\kappa^*([h \partial_{x_i}, h \partial_{x_j}]), \quad \mathbf{\Gamma}_{i\mu}^\kappa = \Upsilon_\kappa^*([h \partial_{x_i}, \Upsilon_\mu]), \quad c_{\lambda\mu}^\kappa = \Upsilon_\kappa^*([\Upsilon_\lambda, \Upsilon_\mu]).$$

Then $\varsigma = \sum_{j=1} dx_i \wedge dp_i$, while

$$\begin{aligned} \tilde{\varsigma} = & \sum_{i=1}^n \pi^* dx_i \wedge da_i + \sum_{\kappa=1}^{\infty} \Upsilon_{\kappa}^* \wedge db_i \\ & + \sum_{\kappa=1}^{\nu} b_{\kappa} \left(\sum_{i,j=1}^n \mathcal{R}_{ij}^{\kappa} \pi^* dx_i \wedge \pi^* dx_j + \sum_{i=1}^n \sum_{\mu=1}^{\nu} \Gamma_{i\mu}^{\kappa} \pi^* dx_i \wedge \Upsilon_{\mu}^* + \sum_{\lambda,\mu=1}^{\nu} c_{\lambda\mu}^{\kappa} \Upsilon_{\lambda}^* \wedge \Upsilon_{\mu}^* \right) \end{aligned}$$

It is then simple to verify that \mathcal{H}^2 is spanned by

$$h^2 \partial_{x_i} = h \partial_{x_i} + \sum_{\kappa,\mu=1}^{\nu} b_{\mu} \Gamma_{i\kappa}^{\mu} \partial_{b_{\kappa}} \quad \text{and} \quad h^2 \partial_{p_i} = \partial_{a_i},$$

or

$$\ell \partial_{x_i} = h \partial_{x_i} - \sum_{\kappa=1}^{\nu} \sum_{j=1}^n b_{\kappa} (\mathcal{R}_{ij}^{\kappa} - \mathcal{R}_{ji}^{\kappa}) \partial_{a_j} + \sum_{\kappa,\mu=1}^{\nu} b_{\mu} \Gamma_{i\kappa}^{\mu} \partial_{b_{\kappa}} \quad \text{and} \quad \ell \partial_{p_i} = \partial_{a_i}.$$

5.2. Proof of Theorem 3(b). We will use the notation of Section 5.1. First, consider any curve $t \mapsto \alpha(t)$ in $\text{Ann}(D)$ which has a projection γ in M . For sufficiently short time, define $\tilde{\alpha}$ as the \mathcal{H}^2 -horizontal lift. Then by Lemma 6, this is in $\text{Ann}(\tilde{D})$. Hence, if $w \in T \text{Ann}(D) \subset TTM$, then $h_p^2 w \in T \text{Ann}(\tilde{D})$.

Let $t \mapsto \tilde{\lambda}(t)$ be a characteristic of \tilde{D} with $\pi^2(\tilde{\lambda}(t)) = \lambda(t)$. We also write $\Pi^{T^*Q}(\tilde{\lambda}) = \tilde{\gamma}$ and $\Pi^{T^*M}(\lambda) = \gamma$. Finally, let $v(t) = \tilde{\lambda}(t) \text{pr}_{\mathcal{V}}$. We then have the following identities that $\tilde{\lambda}(t)$ must satisfy.

For any $\tilde{\alpha}_0 \in \text{Ann}(\tilde{D})_{\tilde{\gamma}(t)}$, we need to have

$$\varsigma(\dot{\tilde{\lambda}}(t), \text{vl}_{\tilde{\lambda}(t)} \alpha_0) = \alpha_0(\dot{\tilde{\gamma}}) = 0,$$

which is the horizontally requirement. Furthermore, let X , be any vector field on M . Then

$$\begin{aligned} \varsigma(\dot{\tilde{\lambda}}, h^2 \mathbf{0}^{T^*M} *_X) &= \varsigma(\dot{\lambda}, \mathbf{0}^{T^*M} *_X) - \tilde{\lambda}(t) \mathcal{R}(X, \dot{\gamma}) \\ &= \varsigma(\dot{\lambda}, \mathbf{0}^{T^*M} *_X) - v(t) \mathcal{R}(X, \dot{\gamma}) = 0 \end{aligned}$$

Finally, for any $\Upsilon \in \Gamma(\mathcal{V})$, we have

$$\tilde{\varsigma}(\dot{\tilde{\lambda}}, \mathbf{0}^{T^*Q} *_\Upsilon) = -d\vartheta^{\mathcal{V}}(\dot{\tilde{\lambda}}, \mathbf{0}^{T^*Q} *_\Upsilon) = -\nabla_{\dot{\gamma}} v(t)(\Upsilon) = 0$$

5.3. Equations for a pendulum with varying length. Consider a pendulum in \mathbb{R}^2 with mass m , under the influence of a force $mA(\cos \phi_0, \sin \phi_0)$. The pendulum is connected to a massless rod or string whose length varies according to the function $\rho(t)$. We give the pendulum coordinates $\rho(t)(\cos \theta(t), \sin \theta(t))$. Newtons equation for this problem is then

$$\begin{aligned} & m(\ddot{\rho} - \rho \dot{\theta}^2)(\cos \theta, \sin \theta) + m(\rho \ddot{\theta} + 2\dot{\rho} \dot{\theta})(-\sin \theta, \cos \theta) \\ &= mA \cos(\theta - \phi_0)(\cos \theta, \sin \theta) - mA \sin(\theta - \phi_0)(-\sin \theta, \cos \theta). \end{aligned}$$

The radial part of this equation is just the equation describing the constraint, while the other part is the essentially equation (10).

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